

Title	Laplace hyperfunctions from the viewpoint of Hodge-Dolbeault cohomology (Microlocal analysis and asymptotic analysis)
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Citation	数理解析研究所講究録 = RIMS Kokyuroku (2019), 2101: 157-164
Issue Date	2019-02
URL	http://hdl.handle.net/2433/251824
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

Laplace hyperfunctions from the viewpoint of Čech-Dolbeault cohomology

By

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Abstract

In this note, we introduce Laplace hyperfunctions from the view point of Čech-Dolbeault cohomology. Furthermore, we construct a Laplace transformation for a Čech-Dolbeault representation of a Laplace hyperfunction.

§ 1. Introduction

Recently, T. Suwa [12] and N. Honda [1] study the theory of Sato's hyperfunctions from the viewpoint of the Čech-Dolbeault cohomology. In their studies, a hyperfunction on \mathbb{R}^n can be represented by a pair (τ_1, τ_{01}) , where τ_1 is a $(0, n)$ -form of C^∞ -coefficients on \mathbb{C}^n and τ_{01} is $(0, n-1)$ -form of a C^∞ -coefficients on $\mathbb{C}^n \setminus \mathbb{R}^n$. The one of advantages for such a presentation is that we can employ, in the theory of hyperfunctions, the similar techniques as those in the C^∞ category such as the partition of unity.

H. Komatsu ([5]-[11]) introduced the theory of Laplace hyperfunctions of one variable in order to consider the Laplace transform of a hyperfunction. The theory of Laplace hyperfunctions in several variables has been established by the author and N. Honda ([3],[4]). As we did in the hyperfunction theory, it is quite natural to study a Laplace hyperfunction from the viewpoint of the Čech-Dolbeault cohomology. In this note, we first describe a Laplace hyperfunction as a pair of C^∞ forms of exponential growth order at ∞ by using Čech-Dolbeault cohomology. Then, we define a Laplace transformation of a Čech-Dolbeault representative of a Laplace hyperfunction and its inverse Laplace

2010 Mathematics Subject Classification(s): Primary 32A45; Secondary 44A10.

Key Words: Laplace transformation, Laplace hyperfunctions.

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transformation in our settings. For details, we refer the reader to the forthcoming paper [2].

At the end of this section, the author would like to express my gratitude to Professor Tatsuo Suwa and Professor Naofumi Honda for the valuable advises.

§ 2. Laplace hyperfunctions of a Čech-Dolbeault representation

Let $n \in \mathbb{N}$ and let M be an n -dimensional \mathbb{R} -vector space, and let E be a complexification $M \otimes_{\mathbb{R}} \mathbb{C}$ of M . We denote by E^\times (resp. M^\times) the set $E \setminus 0$ (resp. $M \setminus 0$) and by \mathbb{R}_+ the set of positive real numbers. Let \mathbb{D}_E (resp. \mathbb{D}_M) be the radial compactification $E \sqcup E^\times/\mathbb{R}_+$ (resp. $M \sqcup M^\times/\mathbb{R}_+$) of E (resp. M). We set $M_\infty := \mathbb{D}_M \setminus M$ and $E_\infty = \mathbb{D}_E \setminus E$. For a subset $T \subset \mathbb{D}_E$, the subset $N_\infty(T)$ in E_∞ is defined by

$$N_\infty(T) := E_\infty \setminus \overline{(E \setminus T)},$$

where the closure is taken in \mathbb{D}_E . Let U be an open subset in E . We also the open subset \widehat{U} in \mathbb{D}_E by

$$\widehat{U} := U \cup N_\infty(U).$$

We sometimes write $\sim U$ instead of \widehat{U} .

Let V be an open subset in \mathbb{D}_E and f a measurable function on $V \cap E$. We say that f is of exponential type (at ∞) on V if, for any compact subset K in V , there exists $H_K > 0$ such that $|\exp(-H_K|z|)f(z)|$ is essentially bounded on $K \cap E$. Let $\mathcal{Q}_{\mathbb{D}_E}(V)$ designate the set of C^∞ functions on $V \cap E$ whose higher derivatives are of exponential type. We denote by $\mathcal{Q}_{\mathbb{D}_E}$ the associated sheaf on \mathbb{D}_E of the presheaf $\{\mathcal{Q}_{\mathbb{D}_E}(V)\}_V$ and by $\mathcal{Q}_{\mathbb{D}_E}^{p,q}$ the sheaf on \mathbb{D}_E of (p, q) -forms with coefficients in $\mathcal{Q}_{\mathbb{D}_E}$. Set

$$\mathcal{Q}_{\mathbb{D}_E}^k = \bigoplus_{p+q=k} \mathcal{Q}_{\mathbb{D}_E}^{p,q}.$$

Now we define the de-Rham complex $\mathcal{Q}_{\mathbb{D}_E}^\bullet$ on \mathbb{D}_E with coefficients in $\mathcal{Q}_{\mathbb{D}_E}$ by

$$0 \longrightarrow \mathcal{Q}_{\mathbb{D}_E}^0 \xrightarrow{d} \mathcal{Q}_{\mathbb{D}_E}^1 \xrightarrow{d} \dots \xrightarrow{d} \mathcal{Q}_{\mathbb{D}_E}^{2n} \longrightarrow 0,$$

and the Dolbeault complex $\mathcal{Q}_{\mathbb{D}_E}^{p,\bullet}$ on \mathbb{D}_E by

$$0 \longrightarrow \mathcal{Q}_{\mathbb{D}_E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{Q}_{\mathbb{D}_E}^{p,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{Q}_{\mathbb{D}_E}^{p,n} \longrightarrow 0.$$

Let $\mathcal{O}_{\mathbb{D}_E}^{\text{exp}}$ denote the sheaf of holomorphic functions of exponential type (at ∞) on \mathbb{D}_E .

We have the following proposition and theorem.

Proposition 2.1. *Both the canonical morphisms of complexes below are quasi-isomorphic:*

$$\mathbb{C}_{\mathbb{D}_E} \longrightarrow \mathcal{Q}_{\mathbb{D}_E}^\bullet, \quad \mathcal{O}_{\mathbb{D}_E}^{\text{exp},(p)} \longrightarrow \mathcal{Q}_{\mathbb{D}_E}^{p,\bullet}.$$

Theorem 2.2. *Assume that $V \cap E$ is Stein and that V is regular at ∞ . Then we have the quasi-isomorphism*

$$\mathcal{O}_{\mathbb{D}_E}^{\text{exp},(p)}(V) \longrightarrow \mathcal{Q}_{\mathbb{D}_E}^{p,\bullet}(V).$$

We have the edge of the wedge theorem for holomorphic functions of exponential type.

Theorem 2.3 ([3], Theorem 3.12). *The complexes $\mathbf{R}\Gamma_{\mathbb{D}_M}(\mathcal{O}_{\mathbb{D}_E}^{\text{exp},(p)})$ and $\mathbf{R}\Gamma_{\mathbb{D}_M}(\mathbb{Z}_{\mathbb{D}_E})$ are concentrated in degree n . Furthermore, $\mathbf{H}_{\mathbb{D}_M}^n(\mathbb{Z}_{\mathbb{D}_E})$ is isomorphic to $\mathbb{Z}_{\mathbb{D}_M}$.*

Define

$$\begin{aligned} \mathcal{B}_{\mathbb{D}_M}^{\text{exp},(p)} &:= \mathbf{H}_{\mathbb{D}_M}^n(\mathcal{O}_{\mathbb{D}_E}^{\text{exp},(p)}) \otimes_{\mathbb{Z}_{\mathbb{D}_M}} \text{or}_{\mathbb{D}_E/\mathbb{D}_M}, \\ \text{or}_{\mathbb{D}_E/\mathbb{D}_M} &:= \mathbf{H}_{\mathbb{D}_M}^n(\mathbb{Z}_{\mathbb{D}_E}). \end{aligned}$$

By Theorem 2.3, we have

$$\mathcal{B}_{\mathbb{D}_M}^{\text{exp},(p)}(\Omega) = \mathbf{H}_{\mathbb{D}_M}^n(V; \mathcal{O}_{\mathbb{D}_E}^{\text{exp},(p)}) \otimes_{\mathbb{Z}_{\mathbb{D}_M}(\Omega)} \mathbf{H}_{\mathbb{D}_M}^n(V; \mathbb{Z}_{\mathbb{D}_E})$$

for any open subset Ω in \mathbb{D}_M . Here V is an open subset in \mathbb{D}_E with $V \cap \mathbb{D}_M = \Omega$.

We can construct the boundary value map in a functorial way.

Theorem 2.4. *Let U be an open subset in \mathbb{D}_E which satisfies $\mathbb{D}_M \subset \overline{U}$. Assume that U is cohomologically trivial in \mathbb{D}_E . Then we have the boundary value map*

$$b_U : \mathcal{O}_{\mathbb{D}_E}^{\text{exp}}(U) \longrightarrow \mathcal{B}_{\mathbb{D}_M}^{\text{exp}}(\mathbb{D}_M).$$

Set $V_0 = \mathbb{D}_E \setminus \mathbb{D}_M$, $V_1 = \mathbb{D}_E$ and $V_{01} = V_0 \cap V_1$. Then define the coverings

$$\mathcal{V} = \{V_0, V_1\}, \quad \mathcal{V}' = \{V_{01}\}.$$

Let $\mathcal{Q}_{\mathbb{D}_E}^{p,\bullet}(\mathcal{V}, \mathcal{V}')$ denote the Čech-Dolbeault complex to the pair $(\mathcal{V}, \mathcal{V}')$ of coverings with coefficients in $\mathcal{Q}_{\mathbb{D}_E}$, i.e.,

$$0 \longrightarrow \mathcal{Q}_{\mathbb{D}_E}^{p,0}(\mathcal{V}, \mathcal{V}') \xrightarrow{\bar{\partial}} \mathcal{Q}_{\mathbb{D}_E}^{p,1}(\mathcal{V}, \mathcal{V}') \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{Q}_{\mathbb{D}_E}^{p,n}(\mathcal{V}, \mathcal{V}') \longrightarrow 0.$$

Here

$$\mathcal{Q}_{\mathbb{D}_E}^{p,k}(\mathcal{V}, \mathcal{V}') = \mathcal{Q}_{\mathbb{D}_E}^{p,k}(V_1) \oplus \mathcal{Q}_{\mathbb{D}_E}^{p,k-1}(V_{01}),$$

$$\bar{\vartheta}(\xi_1, \xi_{01}) = (\bar{\partial}\xi_1, \xi_1|_{V_{01}} - \bar{\partial}\xi_{01}).$$

Let $\mathcal{Q}_{\mathbb{D}_E}^\bullet(\mathcal{V}, \mathcal{V}')$ denote the Čech - de-Rham complex to the pair $(\mathcal{V}, \mathcal{V}')$ of coverings with coefficients in $\mathcal{Q}_{\mathbb{D}_E}$.

$$0 \longrightarrow \mathcal{Q}_{\mathbb{D}_E}^0(\mathcal{V}, \mathcal{V}') \xrightarrow{D} \mathcal{Q}_{\mathbb{D}_E}^1(\mathcal{V}, \mathcal{V}') \xrightarrow{D} \dots \xrightarrow{D} \mathcal{Q}_{\mathbb{D}_E}^{2n}(\mathcal{V}, \mathcal{V}') \longrightarrow 0,$$

where D is used to denote the differential of this complex.

Theorem 2.5. *We have the canonical quasi-isomorphisms:*

$$\mathbf{R}\Gamma_{\mathbb{D}_M}(\mathbb{D}_E; \mathcal{O}_{\mathbb{D}_E}^{\exp.(p)}) \simeq \mathcal{Q}_{\mathbb{D}_E}^{p,\bullet}(\mathcal{V}, \mathcal{V}'), \quad \mathbf{R}\Gamma_{\mathbb{D}_M}(\mathbb{D}_E; \mathbb{C}_{\mathbb{D}_E}) \simeq \mathcal{Q}_{\mathbb{D}_E}^\bullet(\mathcal{V}, \mathcal{V}').$$

In what follows, we constantly use the notations below:

$$\mathbf{H}_{\bar{\vartheta}, \mathcal{Q}}^{p,k}(\mathcal{V}, \mathcal{V}') := \mathbf{H}^k(\mathcal{Q}_{\mathbb{D}_E}^{p,\bullet}(\mathcal{V}, \mathcal{V}')), \quad \mathbf{H}_{D, \mathcal{Q}}^k(\mathcal{V}, \mathcal{V}') := \mathbf{H}^k(\mathcal{Q}_{\mathbb{D}_E}^\bullet(\mathcal{V}, \mathcal{V}')).$$

Hence we have

$$\mathcal{B}_{\mathbb{D}_M}^{\exp.(p)}(\mathbb{D}_M) \simeq \mathbf{H}_{\bar{\vartheta}, \mathcal{Q}}^{p,n}(\mathcal{V}, \mathcal{V}') \otimes_{\mathbb{Z}_{\mathbb{D}_M}(\mathbb{D}_M)} \text{or}_{\mathbb{D}_M/\mathbb{D}_E}(\mathbb{D}_M).$$

§ 3. Laplace transformation

Let $(z_1 = x_1 + \sqrt{-1}y, \dots, z_n = x_n + \sqrt{-1}y_n)$ be the coordinates system of E . We fix the orientation of M and E so that $\{dx_1, dx_2, \dots, dx_n\}$ gives the positive orientation on M , and $\{dy_1, \dots, dy_n, dx_1, \dots, dx_n\}$ give the one on E . Let M^* and E^* be dual vector spaces of M and E respectively. Then we also define the radial compactification \mathbb{D}_{M^*} (resp. \mathbb{D}_{E^*}) of M^* (resp. E^*).

Let Ω be an open subset in \mathbb{D}_{E^*} and f a holomorphic function on Ω . We say that f is of infra-exponential type (at ∞) on Ω if, for any compact set $K \subset \Omega$ and any $\epsilon > 0$, there exists $C > 0$ such that

$$|f(\zeta)| \leq Ce^{\epsilon|\zeta|} \quad (\zeta \in K \cap E^*).$$

Let $\mathcal{O}_{\mathbb{D}_{E^*}}^{\text{inf}}$ designate the set of holomorphic functions of infra-exponential type on \mathbb{D}_{E^*} . and let $\mathcal{A}_{\mathbb{D}_M}^{\text{exp}}$ denote the sheaf of real analytic functions of exponential type. Let $j : M \hookrightarrow \mathbb{D}_M$ be the canonical inclusion. The sheaf $\mathcal{V}_{\mathbb{D}_M}^{\text{exp}}$ of real analytic volumes of exponential type is given by

$$\mathcal{V}_{\mathbb{D}_M}^{\text{exp}} = \mathcal{O}_{\mathbb{D}_E}^{\exp.(n)} \Big|_{\mathbb{D}_M} \otimes_{\mathbb{Z}_{\mathbb{D}_M}} \text{or}_{\mathbb{D}_M}.$$

Here $\text{or}_{\mathbb{D}_M} := j_* \text{or}_M$.

Definition 3.1. Let D be a subset in \mathbb{D}_E and non-zero $\zeta_0 \in E_\infty^*$. We say that D is properly contained in a half space (of \mathbb{D}_E) with direction ζ_0 if there exists a point $a \in E$ such that

$$(3.1) \quad \overline{D} \setminus \{a\} \subset \mathring{\cap} \{z \in E; \operatorname{Re}\langle z - a, \zeta_0 \rangle > 0\}.$$

holds. In the similar way, for $\xi_0 \in M_\infty^*$, we say that a subset D in \mathbb{D}_M is properly contained in a half space (of \mathbb{D}_M) of direction ξ_0 if there exists a point $a \in M$ such that

$$(3.2) \quad \overline{D} \setminus \{a\} \subset \mathring{\cap} \{x \in M; \langle x - a, \xi_0 \rangle > 0\}.$$

Let K be a closed subset in \mathbb{D}_M properly contained in a half space of \mathbb{D}_M with direction $\xi_0 \in M_\infty^*$ and V its open neighborhood of K in \mathbb{D}_E . Set $U := \mathbb{D}_M \cap V$ and

$$\mathcal{V}_K := \{V_0 := V \setminus K, V_1 := V\}, \quad \mathcal{V}'_K := \{V_0\}.$$

Then we have

$$\Gamma_K(U; \mathcal{B}_{\mathbb{D}_M}^{\exp} \otimes_{\mathcal{A}_{\mathbb{D}_M}^{\exp}} \mathcal{V}_{\mathbb{D}_M}^{\exp}) \simeq H_{\overline{\mathcal{V}}, \mathcal{Q}}^{n, n}(\mathcal{V}_K, \mathcal{V}'_K) \otimes_{\mathbb{Z}_{\mathbb{D}_M}(U)} \operatorname{or}_{\mathbb{D}_M/\mathbb{D}_E}(U) \otimes_{\mathbb{Z}_{\mathbb{D}_M}(U)} \operatorname{or}_M(U \cap M).$$

Let $u \otimes a_{\mathbb{D}_M/\mathbb{D}_E} \otimes a_M \in \Gamma_K(U; \mathcal{B}_{\mathbb{D}_M}^{\exp} \otimes_{\mathcal{A}_{\mathbb{D}_M}^{\exp}} \mathcal{V}_{\mathbb{D}_M}^{\exp})$, and let $\nu = (\nu_1, \nu_{01}) \in \mathcal{Q}_{\mathbb{D}_E}^{n, n}(\mathcal{V}_K, \mathcal{V}'_K)$ be a representative of u , i.e., $u = [\nu]$. For this element, we define the Laplace transformation as follows.

Definition 3.2. The Laplace transformation of u is defined by

$$L(u)(\zeta) := \int_D e^{-z\zeta} \nu_1 - \int_{\partial D} e^{-z\zeta} \nu_{01},$$

where D is a contractible open subset in \mathbb{D}_E with (partially) smooth boundary such that $K \subset D \subset \overline{D} \subset V$ and it is properly contained in a half space of \mathbb{D}_E with direction ξ_0 .

Note that $L(u)$ is independent of the choices of a representative ν of u and D of the integral. Let Γ be a proper closed cone in M and $a \in M$. We denote by $\Gamma^\circ \subset E^*$ the dual open cone of Γ in E^* . Assume that $K = \overline{\{a\}} + \overline{\Gamma}$. Then we have the following proposition.

Proposition 3.3. $e^{a\zeta} L(u)$ belongs to $\mathcal{O}_{\mathbb{D}_{E^*}}^{\inf}(N_\infty(\Gamma^\circ))$.

§ 4. Inverse transformation

To construct a inverse Laplace transformation, we prepare some definitions. Let T be a real analytic manifold and set

$$Y := T \times \mathbb{D}_E, \quad Y_\infty = T \times (\mathbb{D}_E \setminus E).$$

We denote by $p_T : Y \rightarrow T$ (resp. $p_{\mathbb{D}_E} : Y \rightarrow \mathbb{D}_E$) the canonical projection to T (resp. \mathbb{D}_E).

Let W be an open subset Y and $f(t, z)$ a measurable function on $W \setminus Y_\infty$. We say that $f(t, z)$ is of exponential type on W if, for any compact subset K in W , there exists $H_K > 0$ such that $|\exp(-H_K|z|)| f(t, z)|$ is essentially bounded on $K \setminus Y_\infty$.

Now we introduce the set $\mathcal{LQ}_Y(W)$ consisting of a measurable function $f(t, z)$ on $W \setminus Y_\infty$ which satisfies the following conditions:

1. For almost every t_0 in $p_T(W)$, $f(t_0, z)$ is a C^∞ function of the variables z on $(p_T^{-1}(t_0) \cap W) \setminus Y_\infty$.
2. Any higher derivative of $f(t, z)$ with respect to the variables z is of exponential type on W .

Let \mathcal{LQ}_Y^k denotes the sheaf on Y of k -forms with respect to the variables in \mathbb{D}_E , and let us define the de-Rham complex \mathcal{LQ}_Y^\bullet by

$$0 \longrightarrow \mathcal{LQ}_Y^0 \xrightarrow{d_{\mathbb{D}_E}} \mathcal{LQ}_Y^1 \xrightarrow{d_{\mathbb{D}_E}} \dots \xrightarrow{d_{\mathbb{D}_E}} \mathcal{LQ}_Y^{2n} \longrightarrow 0,$$

where $d_{\mathbb{D}_E}$ is the differential on \mathbb{D}_E . We denote by $\mathcal{L}_{loc,T}^\infty$ the sheaf of L_{loc}^∞ -functions on T . We have the following two propositions.

Proposition 4.1. *We have the quasi-isomorphism*

$$p_T^{-1} \mathcal{L}_{loc,T}^\infty \longrightarrow \mathcal{LQ}_Y^\bullet.$$

Proposition 4.2. *The complex $\mathbf{R}\Gamma_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(p_T^{-1} \mathcal{L}_{loc,T}^\infty)$ is concentrated in degree n , and we have the canonical isomorphism*

$$\tilde{p}_T^{-1} \mathcal{L}_{loc,T}^\infty \otimes_{\mathbb{Z}_Y} \text{ or } r_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)/Y} \longrightarrow \mathbf{H}_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}^n(p_T^{-1} \mathcal{L}_{loc,T}^\infty),$$

where $\tilde{p}_T : p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M) = T \times \mathbb{D}_M \rightarrow T$ is the canonical projection.

Let Γ be an \mathbb{R}_+ -conic proper open subset in M and $a \in M$.

Let $f \in e^{-a\zeta} \mathcal{O}_{\mathbb{D}_{E^*}}^{\text{inf}}(N_\infty(\Gamma^\circ))$. Then, by the definition, we can easily see:

1. There exists a continuous function $\psi : (N_\infty(\Gamma^\circ) \cap M_\infty^*) \times [0, \infty) \rightarrow \mathbb{R}_{\geq 0}$ such that, for each $\xi_* \in (N_\infty(\Gamma^\circ) \cap M_\infty^*)$, the function $\psi(\xi_*, t)$ is an infra-linear function of the variable t and f is holomorphic on an open subset $W \cap E^*$, where

$$(4.1) \quad W := \bigwedge \left\{ \zeta = t\xi_* + \sqrt{-1}\eta; \eta \in M^* \setminus \{0\}, \xi_* \in (N_\infty(\Gamma^\circ) \cap M_\infty^*), t > \psi(\xi_*, |\eta|) \right\}.$$

Here we identify M_∞^* with $S^{n-1} \subset M^*$.

2. There exists a continuous infra-linear function $\varphi(t)$ on $[0, \infty)$ such that

$$(4.2) \quad |f(\zeta)| \leq e^{-\operatorname{Re}(a\zeta) + \varphi(|\zeta|)} \quad (\zeta \in W \cap E^*).$$

We also define an n -dimensional chain in E^* by

$$\gamma^* := \{\zeta = \xi + \sqrt{-1}\eta \in E^*; \eta \in M^* \setminus \{0\}, \xi = \psi_{\xi_0}(|\eta|)\xi_0\},$$

where ξ_0 is a unit vector in $N_\infty(\Gamma^\circ) \cap M_\infty^*$ and $\psi_{\xi_0}(t)$ is a continuous infra-linear function on $[0, \infty)$ with $\psi_{\xi_0}(t) > \psi(\xi_0, t)$ ($t \in [0, \infty)$). Set $T = S^{n-1}$ and $Y = S^{n-1} \times \mathbb{D}_E$. Define coverings

$$\mathcal{W} = \{W_0 = Y \setminus p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M), W_1 = Y\}, \quad \mathcal{W}' = \{W_0\}.$$

Recall the isomorphisms

$$\begin{aligned} \Gamma(T; \mathcal{L}_{loc, T}^\infty) &= \Gamma(Y; \tilde{p}_T^{-1} \mathcal{L}_{loc, T}^\infty) \\ &\xrightarrow{\sim} H_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}^n(Y; p_T^{-1} \mathcal{L}_{loc, T}^\infty) = H^n(\mathcal{L} \mathcal{Q}_Y^\bullet(\mathcal{W}, \mathcal{W}')), \end{aligned}$$

and set

$$\Omega := \bigwedge \{(\eta, z) \in S^{n-1} \times E; \langle \eta, \operatorname{Im} z \rangle > 0\} \subset Y.$$

Let $j : \Omega \rightarrow Y$ be the canonical open inclusion. Then we can take a special $\omega = (\omega_1, \omega_{01}) \in \mathcal{L} \mathcal{Q}_Y^n(\mathcal{W}, \mathcal{W}')$ satisfying the following conditions:

1. $D_{\mathbb{D}_E} \omega = 0$ and $[\omega]$ is the image of a constant function $1 \in \Gamma(T; \mathcal{L}_{loc, T}^\infty)$ through the above isomorphisms.
2. We have

$$\operatorname{Supp}_{W_1}(\omega_1) \subset \Omega, \quad \operatorname{Supp}_{W_{01}}(\omega_{01}) \subset \Omega.$$

Now we define the inverse Laplace transformation.

Definition 4.3. The inverse Laplace transform L^{-1} is given by

$$\begin{aligned} L_\omega^{-1}(f) &:= \frac{\nu_{\mathbb{D}_M}}{(2\pi\sqrt{-1})^n} \left(\int_{\gamma^*} f(\zeta) \rho(\omega_1) \left(\frac{\eta}{|\eta|}, z \right) e^{\zeta z} d\zeta, \right. \\ &\quad \left. \int_{\gamma^*} f(\zeta) \rho(\omega_{01}) \left(\frac{\eta}{|\eta|}, z \right) e^{\zeta z} d\zeta \right). \end{aligned}$$

Here $\zeta = \xi + \sqrt{-1}\eta$ and $\nu_{\mathbb{D}_M} = dz \otimes a_{\mathbb{D}_M/\mathbb{D}_E} \in \mathcal{V}_{\mathbb{D}_M}^{\exp}(\mathbb{D}_M)$, where $a_{\mathbb{D}_M/\mathbb{D}_E} \in \operatorname{or}_{\mathbb{D}_M/\mathbb{D}_E}(\mathbb{D}_M)$ is determined by the orientation of γ^* through the isomorphism $\operatorname{or}_{\sqrt{-1}M^*} \simeq \operatorname{or}_{\mathbb{D}_M^*/\mathbb{D}_E^*} \simeq \operatorname{or}_{\mathbb{D}_M/\mathbb{D}_E}$.

We have the following proposition.

Proposition 4.4. *We have*

1. *The integration $L_{\omega}^{-1}(f)$ converges and it belongs to $\mathcal{Q}_{\mathbb{D}_E}^{n,n}(\mathcal{V}, \mathcal{V}') \otimes \mathcal{V}_{\mathbb{D}_M}^{\exp}$. Furthermore, $\overline{\partial}(L_{\omega}^{-1}(f)) = 0$ holds.*
2. *$[L_{\omega}^{-1}(f)]$ does not depend on the choices of ω . It is also independent of ξ_0 and ψ_{ξ_0} which appear in the definition of γ^* .*
3. *The support of $[L^{-1}(f)]$ is contained in $K := \overline{\{a\}} + \Gamma \subset \mathbb{D}_M$.*

We have the Laplace inversion formula.

Theorem 4.5. *We have*

$$L \circ L^{-1} = \text{id}, \quad L^{-1} \circ L = \text{id}.$$

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